

DEGENERATE SL_n : REPRESENTATIONS AND FLAG VARIETIES

EVGENY FEIGIN

ABSTRACT. The degenerate Lie group is a semidirect product of the Borel subgroup with the normal abelian unipotent subgroup. We introduce a class of the highest weight representations of the degenerate group of type A, generalizing the PBW-graded representations of the classical group. Following the classical construction of the flag varieties, we consider the closures of the orbits of the abelian unipotent subgroup in the projectivizations of the representations. We show that the degenerate flag varieties \mathcal{F}_n^a and their desingularizations R_n can be obtained via this construction. We prove that the coordinate ring of R_n is isomorphic to the direct sum of duals of the highest weight representations of the degenerate group. In the end, we state several conjectures on the structure of the highest weight representations.

INTRODUCTION

Let \mathfrak{g} be a simple Lie algebra with the Borel subalgebra \mathfrak{b} and let G, B be the corresponding groups. The irreducible highest weight representation of \mathfrak{g} play fundamental role in the algebraic and geometric Lie theory. In particular, the generalized flag varieties for G can be realized as G -orbits inside the projectivizations of these modules. Let \mathfrak{g}^a and G^a be the degenerate Lie algebra and the degenerate Lie group (see [Fe1],[Fe2], [FF], [FFiL], [PY], [Y]). The Lie algebra is a sum of the subalgebra \mathfrak{b} and of the abelian ideal $\mathfrak{g}/\mathfrak{b}$ with the adjoint action of \mathfrak{b} on $\mathfrak{g}/\mathfrak{b}$. The Lie group is the semidirect product of the Borel subgroup with the normal abelian unipotent subgroup $\exp(\mathfrak{g}/\mathfrak{b})$. In this paper we are concerned with the following question: what are the analogues of the finite-dimensional representations of \mathfrak{g} and of the flag varieties in the degenerate situation?

In this paper we only study the type A case, so from now on $\mathfrak{g} = \mathfrak{sl}_n$ and $G = SL_n$. Recall that in this case the fundamental representations V_{ω_k} are labeled by a number $k = 1, \dots, n-1$ and one has $V_{\omega_k} \simeq \Lambda^k(\mathbb{C}^n)$. Now let $\lambda = \sum_{i=1}^{n-1} m_i \omega_i$ be a dominant weight, m_i are non-negative integers. Then the corresponding highest weight representation V_λ sits inside the tensor product of V_{ω_i} 's, where each factor appears exactly m_i times. The image of the embedding is nothing but the \mathfrak{g} -span of the tensor product of highest weight vectors. It has been shown in [FFoL1],[FFoL2], [Fe1] that the degenerate Lie algebra \mathfrak{g}^a naturally acts on the PBW-graded representations V_λ^a (the associated graded to V_λ with respect to the PBW filtration). In addition, V_λ^a

still sits inside the tensor product of $V_{\omega_i}^a$'s in the same way as V_λ in the tensor product of V_{ω_i} 's. The natural question is: are there natural representations of \mathfrak{g}^a different from V_λ^a ? It turns out that this question is important for the study of the PBW-degeneration of flag varieties, see [Fe1], [Fe2], [FF]. Let us recall basic steps here.

Let \mathfrak{n}^- be the nilpotent subalgebra such that $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$. We denote by N^- the corresponding unipotent subgroup of G . Let $(N^-)^a \simeq \exp(\mathfrak{g}/\mathfrak{b})$ be the abelian unipotent subgroup of G^a . This group is isomorphic to the product of $\dim \mathfrak{n}^-$ copies of the group \mathbb{G}_a – the additive group of the field. Let $v_\lambda \in V_\lambda$ be a highest weight vector. Recall that the generalized flag variety \mathcal{F}_λ is defined as the G -orbit of the highest weight line $\mathbb{C}v_\lambda$ in $\mathbb{P}(V_\lambda)$ (see [FH], [Fu], [K]). The corresponding degenerate flag variety is defined as the closure of the $(N^-)^a$ -orbit of $\mathbb{C}v_\lambda$ in $\mathbb{P}(V_\lambda^a)$ (see [Fe1], [Fe2]). For general λ , this is a normal singular projective variety enjoying explicit description in terms of linear algebra. Let us restrict here to the case of regular dominant λ (i.e. all m_i are positive). Then all such flag varieties are isomorphic and we denote the corresponding variety by \mathcal{F}_n^a . \mathcal{F}_n^a can be explicitly realized as the variety of collections $(V_i)_{i=1}^{n-1}$ of subspaces of an n -dimensional space subject to certain conditions. We note that the entries V_i correspond to fundamental weights ω_i .

In [FF] a desingularization R_n for \mathcal{F}_n^a was constructed in terms of linear algebra. These R_n are smooth projective algebraic varieties, which are Bott towers, i.e. can be constructed as successive fibrations with fibers \mathbb{P}^1 . A point in R_n is a collection $(V_{i,j})$ with $1 \leq i \leq j \leq n-1$. This suggests that the "fundamental" representations of the Lie algebra \mathfrak{g}^a are in one-to-one correspondence with the set of positive (non-necessarily simple) roots. In fact, it turns out that to each positive root α one can attach a representation M_α . These representations are highest weight in a sense that M_α is generated from a highest weight vector v_α by the action of the symmetric algebra of \mathfrak{n}^- (for example, for simple roots α_i one has $M_{\alpha_i} \simeq V_{\omega_i}^a$). In addition, R_n can be embedded into the product of $\mathbb{P}(M_\alpha)$ (over all positive roots α) as the closure of the $(N^-)^a$ -orbit through the product of the highest weight lines. Similar to the classical situation, fundamental representations give us a way to construct a large class of \mathfrak{g}^a -modules. Namely, given a collection of non-negative integers $\mathbf{m} = (m_\alpha)$, α – positive root, we consider the \mathfrak{g}^a -module

$$M_{\mathbf{m}} = S^\bullet(\mathfrak{n}^-) \cdot v_{\mathbf{m}} \subset \bigotimes M_{\alpha}^{\otimes m_\alpha}, \quad v_{\mathbf{m}} = \bigotimes v_{\alpha}^{\otimes m_\alpha}.$$

In particular, if $m_\alpha = 0$ for non-simple α , then $M_{\mathbf{m}} \simeq V_\lambda^a$ for $\lambda = \sum m_{\alpha_i} \omega_i$ (α_i are simple roots). We give more examples of such modules and formulate several conjectures concerning the structure of $M_{\mathbf{m}}$.

By definition, for any two collections \mathbf{m}^1 and \mathbf{m}^2 , we have a \mathfrak{g}^a -equivariant embedding $M_{\mathbf{m}^1 + \mathbf{m}^2} \subset M_{\mathbf{m}^1} \otimes M_{\mathbf{m}^2}$, $v_{\mathbf{m}^1 + \mathbf{m}^2} \mapsto v_{\mathbf{m}^1} \otimes v_{\mathbf{m}^2}$. Dualizing, we obtain an algebra $\bigoplus_{\mathbf{m}} M_{\mathbf{m}}^*$. Our main theorem is as follows:

Theorem 0.1. *The coordinate ring of R_n is isomorphic to $\bigoplus_{\mathbf{m}} M_{\mathbf{m}}^*$.*

Our main tool is the explicit form of the Plücker-type relations in the coordinate ring of R_n .

Finally we note that one can naturally attach to a module $M_{\mathbf{m}}$ the corresponding "flag variety". Namely, let $\mathcal{F}(M_{\mathbf{m}}) \subset \mathbb{P}(M_{\mathbf{m}})$ be the closure of the orbit $(N^-)^a \cdot \mathbb{C}v_{\mathbf{m}}$. These are the so-called \mathbb{G}_a -varieties (see [A], [HT]). We note that if all m_{α} are positive, then $\mathcal{F}(M_{\mathbf{m}}) \simeq R_n$ and if all m_{α} but m_{α_i} vanish, then $\mathcal{F}(M_{\mathbf{m}}) \simeq \mathcal{F}_n^a$.

Our paper is organized as follows:

In Section 1 we settle notation and recall main definitions and constructions.

In Section 2 we state our results and provide examples.

Section 3 is devoted to the proofs and in Section 4 several conjectures are stated.

1. NOTATION AND MAIN OBJECTS

1.1. Classical story. Let \mathfrak{g} be a simple Lie algebra with the Cartan decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Let $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$ be the Borel subalgebra. We denote by Φ^+ the set of positive roots of \mathfrak{g} and by $\alpha_1, \dots, \alpha_l \in \Phi^+$ the set of simple roots. We sometimes write $\alpha > 0$ instead of $\alpha \in \Phi^+$. For $\alpha > 0$ we denote by $f_{\alpha} \in \mathfrak{n}^-$ a weight $-\alpha$ element. Thus we have $\mathfrak{n}^- = \bigoplus_{\alpha > 0} \mathbb{C}f_{\alpha}$. Let G, B, N, N^- and T be the Lie groups corresponding to the Lie algebras $\mathfrak{g}, \mathfrak{b}, \mathfrak{n}, \mathfrak{n}^-$ and \mathfrak{h} .

Let $\omega_1, \dots, \omega_l$ be the fundamental weights. The fundamental weights and simple roots are orthogonal with respect to the Killing form (\cdot, \cdot) on \mathfrak{h}^* : $(\omega_i, \alpha_j) = \delta_{i,j}$. A dominant integral weight λ is given by $\sum_{i=1}^l m_i \omega_i$, $m_i \in \mathbb{Z}_{\geq 0}$. For a dominant integral λ let V_{λ} be the finite-dimensional irreducible highest weight \mathfrak{g} -module with highest weight λ and a highest weight vector v_{λ} such that $\mathfrak{n}v_{\lambda} = 0$, $h v_{\lambda} = \lambda(h)v_{\lambda}$ ($h \in \mathfrak{h}$) and $V_{\lambda} = U(\mathfrak{n}^-)v_{\lambda}$.

The (generalized) flag varieties for G are defined as quotient G/P by the parabolic subgroups. These varieties play crucial role in the geometric representation theory. An important feature of the flag varieties is that they can be naturally embedded into the projectivization of the highest weight modules. Namely, let λ be a dominant weight such that the stabilizer of the line $[v_{\lambda}]$ in G is equal to P . Here and below for a vector v in a vector space V we denote by $[v] \in \mathbb{P}(V)$ the line spanned by v . Then one gets the embedding $G/P \subset \mathbb{P}(V_{\lambda})$ as the G -orbit of the highest weight line $[v_{\lambda}]$. For a dominant weight λ we denote by $\mathcal{F}_{\lambda} \subset \mathbb{P}(V_{\lambda})$ the orbit $G[v_{\lambda}]$ of the highest weight line. These are smooth projective algebraic varieties. It is clear that $\mathcal{F}_{\lambda} \simeq \mathcal{F}_{\mu}$ if and only if for all i $(\lambda, \omega_i) = 0$ is equivalent to $(\mu, \omega_i) = 0$.

1.2. Degenerate version. Let \mathfrak{g}^a be the degenerate Lie algebra defined as a direct sum $\mathfrak{b} \oplus \mathfrak{g}/\mathfrak{b}$ of the Borel subalgebra \mathfrak{b} and abelian ideal $\mathfrak{g}/\mathfrak{b}$ (see [Fe1], [Fe2]). The algebra \mathfrak{b} acts on $\mathfrak{g}/\mathfrak{b}$ via the adjoint action. We denote the space $\mathfrak{g}/\mathfrak{b}$ by $(\mathfrak{n}^-)^a$ (a is for abelian). Let $(N^-)^a = \exp(\mathfrak{n}^-)^a$ be the

abelian Lie group, which is nothing but the product of $\dim \mathfrak{n}^-$ copies of the group \mathbb{G}_a – the additive group of the field. Let G^a be the semidirect product $B \ltimes (N^-)^a$ of the subgroup B and of the normal abelian subgroup $(N^-)^a$ (the action of B on $(N^-)^a$ is induced by the action of B on \mathfrak{n}^- by conjugation).

Similar to the classical situation, we say that a \mathfrak{g}^a -module M is a highest weight module if there exists $v \in M$ such that $\mathfrak{n}v = 0$, the line $[v]$ is \mathfrak{h} -stable and $M = S(\mathfrak{n}^-)v$, where $S(\mathfrak{n}^-)$ denotes the symmetric algebra of \mathfrak{n}^- , which is isomorphic to the polynomial ring $\mathbb{C}[f_\alpha]_{\alpha > 0}$. It is clear that the highest weight \mathfrak{g}^a -modules are in one-to-one correspondence with the \mathfrak{b} -invariant ideals in $\mathbb{C}[f_\alpha]_{\alpha > 0}$. Namely, M defines the annihilating ideal $\text{Ann}M \subset \mathbb{C}[f_\alpha]_{\alpha > 0}$ and a \mathfrak{b} -invariant ideal I produces the \mathfrak{g}^a -module $\mathbb{C}[f_\alpha]_{\alpha > 0}/I$. In [FFoL1], [FFoL2] the modules V_λ^a were studied for dominant integral λ . The module V_λ^a is the associated graded module to V_λ with respect to the PBW filtration. We note that all V_λ^a are highest weight modules. However, these are not all highest weight \mathfrak{g}^a -modules. Below we study a wider class of representations for $\mathfrak{g} = \mathfrak{sl}_n$ (still not exhaustive). We note that all the highest weight \mathfrak{g}^a -modules which show up so far, are graded compatibly with the grading by the total degree on $\mathbb{C}[f_\alpha]_{\alpha > 0}$. Summarizing, we put forward the following definition:

Definition 1.1. A \mathfrak{g}^a -module $M = \bigoplus_{s \geq 0} M(s)$ is called a graded highest weight module if the following holds:

- $M(0) = \mathbb{C}v, \mathfrak{n}v = 0, \mathfrak{h}v \subset M(0)$,
- $M = \mathbb{C}[f_\alpha]_{\alpha > 0}v$,
- $f_\alpha M(s) \subset M(s+1), \alpha > 0$.

We note that the definition implies that $\mathfrak{b}M(s) \subset M(s)$. In what follows we say that elements of $M(s)$ have PBW-degree s .

Now we want to define degenerate version of the flag varieties. It is not reasonable to take the quotient of G^a now (for example, G^a/B is simply an affine space). Instead we use the highest weight representations. Let M be a graded highest weight \mathfrak{g}^a -module with a highest weight vector v . Then we put forward the following definition:

Definition 1.2. The variety $\mathcal{F}(M)$ is the closure of the $(N^-)^a$ -orbit of the highest weight line:

$$\mathcal{F}(M) = \overline{(N^-)^a \cdot [v]} \subset \mathbb{P}(M).$$

The degenerate flag varieties \mathcal{F}_λ^a are isomorphic to $\mathcal{F}(V_\lambda^a)$. We note that the group $(N^-)^a$ is isomorphic to the product of $\dim \mathfrak{n}^-$ copies of the group \mathbb{G}_a – the additive group of the field. Therefore, all the varieties $\mathcal{F}(M)$ are the so-called \mathbb{G}_a -varieties, i.e. the compactifications of the abelian unipotent group (see [A], [HT]).

1.3. The type A case. From now on $\mathfrak{g} = \mathfrak{sl}_n$ and $G = SL_n$. In this case the set of positive roots Φ^+ consists of the roots

$$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \quad 1 \leq i \leq j \leq n-1.$$

In what follows we use the shorthand notation $f_{i,j}$ for $f_{\alpha_{i,j}}$. The following proposition is proved in [FFoL1] (\circ denotes the adjoint action of \mathfrak{b} on $S(\mathfrak{n}^-) \simeq S(\mathfrak{g}/\mathfrak{b})$).

Proposition 1.3. *For a dominant weight λ the module V_λ^a is isomorphic to the quotient $S(\mathfrak{n}^-)/I_\lambda$, where I_λ is the ideal generated by the subspace*

$$U(\mathfrak{b}) \circ \text{span}(f_\beta^{(\beta,\lambda)+1}, \beta \in \Psi^+).$$

It is proved in [Fe2] that the degenerate flag varieties enjoy the following explicit realization. For simplicity, we describe the case of the complete flags here. So let λ be a regular dominant weight, i.e. $(\lambda, \omega_i) > 0$ for all i . Fix an n -dimensional vector space W and a basis w_1, \dots, w_n of W . The operators $f_{i,j}$ act on W by the standard formula $f_{i,j}w_k = \delta_{i,k}w_{j+1}$. Let $pr_k : W \rightarrow W$ be a projection along the k -th basis vector, i.e. $pr_kw_k = 0$ and $pr_kw_j = w_j$ for $j \neq k$. All the flag varieties \mathcal{F}_λ^a are isomorphic (λ is regular) and we denote this variety by \mathcal{F}_n^a . Then \mathcal{F}_n^a consists of collections (V_1, \dots, V_{n-1}) of subspaces of W such that $\dim V_i = i$ and $pr_{i+1}V_i \subset V_{i+1}$. The varieties \mathcal{F}_n^a are singular (starting from $n = 3$) projective algebraic varieties, naturally embedded into the product of Grassmannians. A desingularization R_n for \mathcal{F}_n^a was constructed in [FF] (see also [FFiL] for the symplectic version). Namely, R_n consists of collections $(V_{i,j})_{1 \leq i \leq j \leq n-1}$ of subspaces of W subject to the following conditions:

- $\dim V_{i,j} = i$, $V_{i,j} \subset \text{span}(w_1, \dots, w_i, w_{j+1}, \dots, w_n)$,
- $V_{i,j} \subset V_{i+1,j}$, $1 \leq i < j \leq n-1$,
- $pr_{j+1}V_{i,j} \subset V_{i,j+1}$, $1 \leq i \leq j < n-1$.

One can show that the variety R_n is a Bott tower, i.e. there exists a tower

$$R_n = R_n(0) \rightarrow R_n(1) \rightarrow R_n(2) \rightarrow \cdots \rightarrow R_n(n(n-1)/2) = pt$$

such that each map $R_n(l) \rightarrow R_n(l+1)$ is a \mathbb{P}^1 -fibration. Hence, R_n is smooth. The surjection $R_n \rightarrow \mathcal{F}_n^a$ is given by $(V_{i,j})_{1 \leq i \leq j \leq n-1} \mapsto (V_{i,i})_{i=1}^n$. Let $W_{i,j} = \text{span}(w_1, \dots, w_i, w_{j+1}, \dots, w_n)$. Then from the definition of R_n we obtain

$$(1.1) \quad R_n \subset \prod_{1 \leq i \leq j \leq n-1} \mathbb{P}(\Lambda^i W_{i,j}).$$

2. REPRESENTATIONS AND COORDINATE RINGS

In this section we state our results and provide examples. The proofs are given in the next section. Throughout the section $\mathfrak{g} = \mathfrak{sl}_n$.

2.1. Representations. For a positive root $\alpha = \alpha_{i,j}$ let M_α be the \mathfrak{g}^a -module defined as the quotient $\mathbb{C}[f_\beta]_{\beta > 0}/I_\alpha$, where I_α is the ideal generated by the subspace

$$U(\mathfrak{b}) \circ \langle f_\beta^2, \beta \geq \alpha; f_\beta, \beta \not\geq \alpha \rangle.$$

(Here \circ denotes the adjoint action). We denote the highest weight vector (the image of 1) in M_α by v_α .

Example 2.1. Let $\alpha = \alpha_1 + \dots + \alpha_{n-1}$ be the highest root. Then M_α is two-dimensional space with a basis $v_\alpha, f_\alpha v_\alpha$. All the elements $f_\beta, \beta \neq \alpha$ as well as $\mathfrak{b} \subset \mathfrak{g}^a$, act trivially on M_α .

Lemma 2.2. *If α is a simple root α_i , then $M_{\alpha_i} \simeq V_{\omega_i}^a$ as \mathfrak{g}^a -modules.*

Proof. Follows from Proposition 1.3. \square

Lemma 2.3. *Let $\alpha = \alpha_{i,j}$. Then $M_\alpha \simeq \Lambda^i W_{i,j}$.*

Proof. Let us define the structure of \mathfrak{g}^a -module on $\Lambda^i W_{i,j}$ in the following way. Consider $V_{\omega_i}^a$. Then as a vector space this module is isomorphic to $\Lambda^i W$. It is easy to see that the subspace U spanned by the vectors

$$w_{l_1} \wedge \dots \wedge w_{l_i}, \exists r : i < l_r \leq j$$

is \mathfrak{g}^a -invariant. Now clearly we can identify the quotient of $\Lambda^i W/U$ with $\Lambda^i W_{i,j}$. In addition, this quotient is isomorphic to $M_{\alpha_{i,j}}$ according to Proposition 1.3. \square

Remark 2.4. 1). We note that $M_\alpha = \bigoplus_{s \geq 0} M_\alpha(s)$ (see Definition 1.1) and $M_\alpha(s)$ is spanned by the vectors

$$w_{l_1} \wedge \dots \wedge w_{l_i}, l_a \in \{1, \dots, i, j+1, \dots, n\}, \#\{a : l_a > j\} = s.$$

2). The operators $f_{a,b}$ act trivially on $M_{\alpha_{i,j}}$ unless $a \leq i \leq j \leq b$.

The modules $M_{\alpha_{i,j}}$ play a role of the fundamental representations for the Lie algebra \mathfrak{g}^a . We now define a larger class of representations "generated" by the fundamental ones. Let $\mathbf{m} = (m_{i,j})_{i \leq j}$ be a collection of non-negative integers. Here and in what follows we write $i \leq j$ (or $a \leq b$) instead of $1 \leq i \leq j \leq n-1$ (or $1 \leq a \leq b \leq n-1$) if it does not lead to any confusion. We put forward the following definition

Definition 2.5. The \mathfrak{g}^a -module $M_{\mathbf{m}}$ is the top (Cartan) component in the tensor product of modules $M_{\alpha_{i,j}}$, i.e.:

$$M_{\mathbf{m}} = S(\mathfrak{n}^-) \cdot \bigotimes_{i \leq j} v_{\alpha_{i,j}}^{\otimes m_{i,j}} \subset \bigotimes_{i \leq j} M_{\alpha_{i,j}}^{\otimes m_{i,j}}.$$

We denote the highest weight vector of $M_{\mathbf{m}}$, which is the tensor product of the vectors $v_{\alpha_{i,j}}$, by $v_{\mathbf{m}}$.

Example 2.6. Let $m_{i,j} = 0$ unless $i = j$. Then according to [FFoL1] we have $M_{\mathbf{m}} \simeq V_\lambda^a$, where $\lambda = \sum_{i=1}^{n-1} m_{i,i} \omega_i$.

Example 2.7. Let $m_{1,j} = 1$ for $j = 1, 2, \dots, n-1$ and $m_{i,j} = 0$ if $i > 1$. Then $f_{i,j} v_{\mathbf{m}} = 0$ unless $i = 1$ and it is easy to see that $f_{1,1}^{k_1} \dots f_{1,n-1}^{k_{n-1}} v_{\mathbf{m}} \neq 0$ if and only if

$$(2.1) \quad k_1 \leq 1, \quad k_1 + k_2 \leq 2, \dots, k_1 + \dots + k_{n-1} \leq n-1.$$

Moreover, the monomials satisfying conditions above are linearly independent. We note that the number of collections (k_1, \dots, k_{n-1}) subject to the restrictions (2.1) is equal to the Catalan number C_n . In addition, the dimensions of the homogeneous components $\dim M_{\mathbf{m}}(s)$ are given by the entries of the Catalan triangle (see e.g. <http://mathworld.wolfram.com/CatalansTriangle.html>).

The following Lemma explains the importance of the above defined modules. We say that a collection \mathbf{m} is regular if $m_{i,j} > 0$ for all $i \leq j$.

Proposition 2.8. *For any regular \mathbf{m} , the variety R_n is embedded into the projective space $\mathbb{P}(M_{\mathbf{m}})$ as the closure of the SL_n^a -orbit or $(N^-)^a$ -orbit through the highest weight line:*

$$R_n = \overline{SL_n^a \cdot [v_{\mathbf{m}}]} = \overline{(N^-)^a \cdot [v_{\mathbf{m}}]} \subset \mathbb{P}(M_{\mathbf{m}}).$$

Proof. It suffices to show that our proposition holds if all $m_{i,j} = 1$. We note that the orbits $SL_n^a \cdot [v_{\mathbf{m}}]$ and $(N^-)^a \cdot [v_{\mathbf{m}}]$ coincide and are embedded into the product of Grassmannians $\prod_{i \leq j} Gr_i(W_{i,j})$. Recall that we have fixed a basis w_1, \dots, w_n in W . For a collection

$$L = (l_1, \dots, l_i) \subset \{1, \dots, i, j+1, \dots, n\}$$

and an element $V_{i,j} \in Gr_i(W_{i,j})$, let $X_L(V_{i,j})$ be the Plücker coordinate of $V_{i,j}$. We claim that the orbit $(N^-)^a [v_{\mathbf{m}}]$ consists exactly of the collections of subspaces $(V_{i,j})_{i \leq j} \in R_n$ such that $X_{1,\dots,i}(V_{i,j}) \neq 0$ (see Lemma 2.11 and Lemma 2.12 of [FF]). This implies the proposition. \square

In what follows we consider R_n as being embedded into $\mathbb{P}(M_{\mathbf{m}})$ with $m_{i,j} = 1$ for all $i \leq j$, or, equivalently, being embedded into $\prod_{\alpha > 0} \mathbb{P}(M_{\alpha})$. We denote such a collection by $\mathbf{1}$.

2.2. Coordinate ring. We denote by $X_L^{(i,j)} = X_{l_1, \dots, l_i}^{(i,j)}$ the dual (Plücker) coordinates in $W_{i,j}$ with respect to $w_{l_1} \wedge \dots \wedge w_{l_i}$, where $1 \leq l_1 < \dots < l_i \leq n$ and $L \subset \{1, \dots, i, j+1, \dots, n\}$. For a permutation $\sigma \in S_n$ we set

$$X_{l_{\sigma(1)}, \dots, l_{\sigma(i)}}^{(i,j)} = (-1)^{\sigma} X_{l_1, \dots, l_i}^{(i,j)}.$$

The PBW-degree of a variable $X_L^{(i,j)}$ is defined as the number of entries l_a such that $l_a > j$.

Remark 2.9. The variables of the PBW-degree zero are of the form $X_{1, \dots, i}^{(i,j)}$. The variables of PBW-degree one are of the form $X_{1, \dots, r-1, r+1, \dots, i, m}^{(i,j)}$ for some $r \leq i \leq j < m$.

Let us give explicit description of the SL_n^a -orbit through the highest weight line in M_1 . For a collection of complex numbers $(c_{a,b})_{a \leq b}$ consider the element

$$\exp\left(\sum_{a \leq b} c_{a,b} f_{a,b}\right) \in (N^-)^a.$$

Let $L = (1 \leq l_1 < \dots < l_i \leq n)$ and let d be a number such that $l_d \leq i < l_{d+1}$. We consider the numbers $1 \leq r_1 < \dots < r_{i-d} \leq i$ such that

$$\{r_1, \dots, r_{i-d}\} \cup \{l_1, \dots, l_d\} = \{1, \dots, i\}.$$

Lemma 2.10. *For $L \subset \{1, \dots, i, j+1, \dots, n\}$, the Plücker coordinate $X_L^{(i,j)}$ of the point $\exp(\sum_{a \leq b} c_{a,b} f_{a,b})[v_1]$ is given by*

$$(-1)^{\sum_{p=1}^d (l_p - p)} \det(c_{r_a, l_{d+b-1}})_{a,b=1}^{i-d}.$$

Proof. We note that $f_{a,b} w_c = \delta_{a,c} w_{b+1}$. Therefore in $W_{i,j}$ one has:

$$\begin{aligned} & \exp\left(\sum_{a \leq b} c_{a,b} f_{a,b}\right) w_1 \wedge \dots \wedge w_i \\ &= (w_1 + c_{1,j} w_{j+1} + \dots + c_{1,n-1} w_n) \wedge \dots \wedge (w_i + c_{i,j} w_{j+1} + \dots + c_{i,n} w_n). \end{aligned}$$

□

Example 2.11. Let $d = i-1$, i.e. $l_{i-1} \leq i$ and $l_i > j$. Let $\{r\} = \{1, \dots, i\} \setminus L$. Then

$$(2.2) \quad X_L^{(i,j)} \left(\exp\left(\sum_{a \leq b} c_{a,b} f_{a,b}\right)[v_1] \right) = (-1)^{i-r} c_{r, l_{i-1}}.$$

Let $\mathbb{C}\mathbf{X}_n$ be the ring of multi-homogeneous polynomials in variables $X_L^{(i,j)}$, $1 \leq i \leq j \leq n-1$, $L = (l_1, \dots, l_i)$. In other words, $\mathbb{C}\mathbf{X}_n$ is the coordinate ring of the product of projective spaces $\mathbb{P}(W_{i,j})$, $1 \leq i \leq j \leq n-1$. We fix the decomposition

$$\mathbb{C}\mathbf{X}_n = \bigoplus_{\mathbf{m} \in \mathbb{Z}^{n(n-1)/2}} \mathbb{C}\mathbf{X}_{\mathbf{m}},$$

where $\mathbb{C}\mathbf{X}_{\mathbf{m}}$ is the space of polynomials of total degree $m_{i,j}$ in all variables $X_L^{(i,j)}$. For an ideal $I \subset \mathbb{C}\mathbf{X}_n$ we denote by $X(I) \subset \prod_{i \leq j} \mathbb{P}(\Lambda^i W_{i,j})$ the variety of zeros of I and for a subvariety X in the product of projective spaces we denote by $I(X)$ the ideal of multi-homogeneous polynomials vanishing on X .

Our goal is to find the ideal $I(R_n)$ in $\mathbb{C}\mathbf{X}_n$ of multi-homogeneous polynomials vanishing on R_n . Then the coordinate ring \mathcal{Q}_n of R_n is given by

$$\mathcal{Q}_n = \mathbb{C}\mathbf{X}_n / I(R_n).$$

We first describe the analogues of the Plücker relations contained in $I(R_n)$.

The relations are labeled by the following data:

- two pairs $(i_1, j_1), (i_2, j_2)$ with $i_1 \geq i_2$ and $1 \leq i_1 \leq j_1 \leq n-1$, $1 \leq i_2 \leq j_2 \leq n-1$;
- an integer k satisfying $1 \leq k \leq i_2$;
- two collections of indices $L = (l_1, \dots, l_{i_1})$, $P = (p_1, \dots, p_{i_2})$ with $L \subset \{1, \dots, i_1, j_1+1, \dots, n\}$, $P \subset \{1, \dots, i_2, j_2+1, \dots, n\}$.

We denote the relation labeled by the data above by $R_{L,P}^{(i_1,j_1),(i_2,j_2);k}$. This is a polynomial in the ring $\mathbb{C}\mathbf{X}_n$ given by the following formulas:

First, let $i_1 \neq i_2$ and $j_1 \geq j_2$. Then

$$R_{L,P}^{(i_1,j_1),(i_2,j_2);k} = X_L^{(i_1,j_1)} X_P^{(i_2,j_2)} - \sum_{1 \leq r_1 < \dots < r_k \leq n} X_{L'}^{(i_1,j_1)} X_{P'}^{(i_2,j_2)},$$

where the summation runs over r_i satisfying $l_{r_i} \in \{1, \dots, i_2, j_1+1, \dots, n\}$ and for a term $X_{L'}^{(i_1,j_1)} X_{P'}^{(i_2,j_2)}$ corresponding to the sequence (r_1, \dots, r_k) one has

$$(2.3) \quad L' = (l_1, \dots, l_{r_1-1}, p_1, l_{r_1+1}, \dots, l_{r_k-1}, p_k, l_{r_k+1}, \dots, l_{i_1}),$$

$$(2.4) \quad P' = (l_{r_1}, \dots, l_{r_k}, p_{k+1}, \dots, p_{i_2}).$$

Second, let $j_1 \leq j_2$. Then we take the relations as above with an additional restriction $L \subset \{1, \dots, i_1, j_2+1, \dots, n\}$.

Remark 2.12. We note that the initial term $X_L^{(i_1,j_1)} X_J^{(i_2,j_2)}$ is present in the relations if and only if $\{p_1, \dots, p_k\} \subset \{1, \dots, i_2, j_1+1, \dots, n\}$.

Example 2.13. Let $n = 3$. Then $\mathbb{C}\mathbf{X}_3$ the ring of multi-homogeneous polynomials in three groups of variables

$$X_1^{(1,1)}, X_2^{(1,1)}, X_3^{(1,1)}; \quad X_1^{(1,2)}, X_3^{(1,2)}; \quad X_{1,2}^{(2,2)}, X_{1,3}^{(2,2)}, X_{2,3}^{(2,2)}.$$

The Plücker relations are given by

$$R_{(3),(1)}^{(1,1),(1,2);1} = X_3^{(1,1)} X_1^{(1,2)} - X_1^{(1,1)} X_3^{(1,2)},$$

$$R_{(2,3),(1)}^{(2,2),(1,1);1} = X_{2,3}^{(2,2)} X_1^{(1,1)} + X_{1,2}^{(2,2)} X_3^{(1,1)},$$

$$R_{(2,3),(1)}^{(2,2),(1,2);1} = X_{2,3}^{(2,2)} X_1^{(1,2)} + X_{1,2}^{(2,2)} X_3^{(1,2)}.$$

Example 2.14. Let $n = 4$. Then the only relation containing variables $X_{l_1,l_2}^{(2,2)}$ and $X_p^{(1,3)}$ is

$$R_{(1,2),(4)}^{(2,2),(1,3);1} = X_{1,2}^{(2,2)} X_4^{(1,3)} + X_{2,4}^{(2,2)} X_1^{(1,3)}.$$

Example 2.15. Fix a pair $1 \leq i \leq j \leq n-1$. Then the ideal in $\mathbb{C}[X_{l_1, \dots, l_i}^{(i,j)}]$ generated by the relations $R_{L,P}^{(i,j),(i,j);k}$ is exactly the ideal defining the Grassmann variety $Gr_i(W_{i,j})$.

Proposition 2.16. *The polynomials $R_{L,P}^{(i_1,j_1),(i_2,j_2);k}$ vanish on R_n .*

Let $J_n \subset \mathbb{C}\mathbf{X}_n$ be the ideal generated by the Plücker relations above. We will prove the following theorem.

Theorem 2.17. *The ideal $I(R_n)$ is the saturation of J_n . More precisely, for any $F \in I(R_n)$ there exist non-negative integers $N_{i,j}$, $i \leq j$, such that*

$$\prod_{i \leq j} \left(X_{1, \dots, i}^{(i,j)} \right)^{N_{i,j}} F \in J_n.$$

We put forward the following conjecture.

Conjecture 2.18. The ideal J_n is prime, i.e. $I(R_n) = J_n$.

3. PROOFS.

3.1. The ideal.

Proposition 3.1. *$J_n \subset I(R_n)$, i.e. all the relations $R_{L,P}^{(i_1,j_1),(i_2,j_2),k}$ vanish on R_n .*

Proof. We recall that the Sylvester identity (see e.g. [Fu], section 8.1, Lemma 2) states that for any two s by s matrices M and N and a number $k \leq s$ one has

$$\det M \det N - \sum_{1 \leq r_1 < \dots < r_k \leq s} \det M' \det N' = 0,$$

where the matrices M' and N' are obtained from M and N by interchanging the first k columns of N with the columns r_1, \dots, r_k of M (preserving the order of columns). Combining the Sylvester identity with Lemma 2.10 we obtain our proposition. \square

Lemma 3.2. *The relations $R_{L,P}^{(i,j),(i,j);k}$, $R_{L,P}^{(i+1,j),(i+1,j);k}$ and $R_{L,P}^{(i,j),(i+1,j);k}$ with all possible L and P define the variety of pairs $(V_{i,j}, V_{i+1,j}) \subset \text{Gr}_i(W_{i,j}) \times \text{Gr}_{i+1}(W_{i+1,j})$ satisfying $V_{i,j} \subset V_{i+1,j}$.*

Proof. First, the relations $R_{L,P}^{(i,j),(i,j);k}$ and $R_{L,P}^{(i+1,j),(i+1,j);k}$ cut out the product of Grassmann varieties $\text{Gr}_i(W_{i,j}) \times \text{Gr}_{i+1}(W_{i+1,j})$ inside the product of projective spaces $\mathbb{P}(\Lambda^i W_{i,j}) \times \mathbb{P}(\Lambda^{i+1} W_{i+1,j})$. Second, the relations $R_{L,P}^{(i,j),(i+1,j);k}$ are exactly the classical Plücker relations saying that the first subspace has to be a subset of the second. \square

Lemma 3.3. *The relations $R_{L,P}^{(i,j),(i,j);k}$, $R_{L,P}^{(i,j+1),(i,j+1);k}$ and $R_{L,P}^{(i,j),(i,j+1);k}$ with arbitrary L and P define the variety of pairs $(V_{i,j}, V_{i,j+1}) \subset \text{Gr}_i(W_{i,j}) \times \text{Gr}_i(W_{i,j+1})$ satisfying $\text{pr}_{j+1} V_{i,j} \subset V_{i,j+1}$.*

Proof. Similar to the proof of the lemma above. The only difference that the relations $R_{L,P}^{(i,j),(i,j+1);k}$ are degenerate Plücker relations (see [Fe1]). \square

Proposition 3.4. *We have $R_n = X(J_n)$, i.e. the common zeroes of J_n give R_n .*

Proof. We need to prove that if all the relations R vanish at a point $x \in \times_{i \leq j} \mathbb{P}(M_{\alpha_{i,j}})$, then $x \in R_n$. Follows from Lemmas 3.2 and 3.3. \square

The following proposition is of the key importance for us.

Proposition 3.5. *For an element $F \in \mathbb{C}\mathbf{X}_n$ there exist numbers $N_{i,j} \in \mathbb{Z}_{\geq 0}$, $i \leq j$ and another polynomial $G \in \mathbb{C}\mathbf{X}_n$ such that*

$$\prod_{i \leq j} (X_{1,\dots,i}^{(i,j)})^{N_{i,j}} F - G \in J_n$$

and G depends only on the PBW-degree zero or one variables

$$X_{1,\dots,i}^{(i,j)}, \quad i \leq j \quad \text{and} \quad X_{1,\dots,r-1,r+1,\dots,i,m}^{(i,j)}, \quad 1 \leq r \leq i \leq j < m,$$

In addition, G can be chosen in such a way that if G depends on a variable $X_{1,\dots,r-1,r+1,\dots,i,m}^{(i,j)}$ for some i, j then it does not depend on all variables of the form $X_{1,\dots,r-1,r+1,\dots,i_1,m}^{(i_1,j_1)}$ for $(i_1, j_1) \neq (i, j)$.

Proof. Recall the PBW-degree of a variable $X_L^{(i,j)}$ given by the number of $l_a \in L$ such that $l_a > j$. Let $X_L^{(i,j)}$ be a variable of PBW-degree greater than one with $r \notin L$ for some $r \leq i$. We consider the relation $R_{L,(r,1,\dots,r-1,r+1,\dots,i)}^{(i,j),(i,j);1}$. One has

$$\begin{aligned} (-1)^{r-1} R_{L,(1,\dots,i)}^{(i,j),(i,j);1} = \\ X_L^{(i,j)} X_{(1,\dots,i)}^{(i,j)} - \sum_{a: l_a > j} X_{(l_1,\dots,l_{a-1},r,l_{a+1},\dots,l_i)}^{(i,j)} X_{(1,\dots,r-1,l_a,r+1,\dots,i)}^{(i,j)}. \end{aligned}$$

We note that the PBW-degree of each variable $X_{(l_1,\dots,l_{a-1},r,l_{a+1},\dots,l_i)}^{(i,j)}$ is one less than that of $X_L^{(i,j)}$. Hence, by decreasing induction, we obtain the first statement of our proposition.

Now assume that a polynomial G depends on the variables

$$X_{1,\dots,r-1,r+1,\dots,i_1,m}^{(i_1,j_1)} \quad \text{and} \quad X_{1,\dots,r-1,r+1,\dots,i_2,m}^{(i_2,j_2)}$$

for two different pairs (i_1, j_1) and (i_2, j_2) . Let $i_1 \geq i_2$. We consider the relation

$$\begin{aligned} (-1)^{r-1} R_{(1,\dots,r-1,r+1,\dots,i_1,m),(r,1,\dots,r-1,r+1,\dots,i_2)}^{(i_1,j_1),(i_2,j_2);1} \\ = X_{1,\dots,r-1,r+1,\dots,i_1,m}^{(i_1,j_1)} X_{1,\dots,i_2}^{(i_2,j_2)} - (-1)^{i_2+i_1} X_{1,\dots,i_1}^{(i_1,j_1)} X_{1,\dots,r-1,r+1,\dots,i_2,m}^{(i_2,j_2)}. \end{aligned}$$

Using these relations we can get rid of the variable $X_{1,\dots,r-1,r+1,\dots,i_1,m}^{(i_1,j_1)}$ (multiplying by $X_{1,\dots,i_2}^{(i_2,j_2)}$). \square

3.2. Representation of Q_n . Let $Q_n = \mathbb{C}\mathbf{X}_n / I(R_n)$ be the coordinate ring of R_n . Consider a polynomial algebra A_n in variables $T_{i,j}$, $1 \leq i \leq j \leq n-1$ and $Z_{i,j}$, $1 \leq i \leq j \leq n-1$. We define a homomorphism $\Psi : \mathbb{C}\mathbf{X}_n \rightarrow A$ in the following way. Let $X_L^{(i,j)}$ be a variable with

$$1 \leq l_1 < \dots < l_d \leq i \leq j < l_{d+1} < \dots < l_i$$

(thus the PBW-degree of this variable is $i - d$). We define $i \times i$ matrix M by

$$(3.1) \quad M_{a,b} = \begin{cases} (-1)^{l_a - a}, & \text{if } b = a \leq d, l_a \in L \\ Z_{a, l_b - 1}, & \text{if } a \notin L, a \leq i, b > d, \\ 0, & \text{otherwise.} \end{cases}$$

We define the homomorphism of polynomial algebras by the formula

$$(3.2) \quad X_{l_1, \dots, l_i}^{(i,j)} \mapsto T_{i,j} \det M.$$

Proposition 3.6. *The kernel of Ψ coincides with $I(R_n)$.*

Proof. Recall the open dense cell $U \subset R_n$, which is the $(N^-)^a$ -orbit through the highest weight line. We note that $I(R_n) = I(U)$. Now the embedding

$$R_n \subset \prod_{i \leq j} \mathbb{P}(\Lambda^i W_{i,j})$$

is defined using the same determinants as in (3.1) (see Lemma (2.10)). Hence a polynomial in $\mathbb{C}\mathbf{X}_n$ vanishes on U if and only if belongs to the kernel of Ψ (we note that the variables $T_{i,j}$ guarantee the multi-homogeneity). \square

Corollary 3.7. *We have an embedding $Q_n \rightarrow A_n$ defined by (3.2).*

Corollary 3.8. *One has the decomposition*

$$Q_n = \bigoplus_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{n(n-1)/2}} Q_{\mathbf{m}},$$

where $Q_{\mathbf{m}}$ consists of polynomials of total degree $m_{i,j}$ in variables $X_L^{(i,j)}$.

We note that this decomposition is very similar to the one for the classical flag varieties, where the coordinate ring decomposes into the direct sum of dual irreducible \mathfrak{g} -modules. Thus it is very natural to ask about the properties of the spaces $Q_{\mathbf{m}}$.

3.3. Cocyclicity. Our goal in this subsection is to prove that there exists a natural action of the Lie algebra $(\mathfrak{n}^-)^a$ on Q_n such that each $Q_{\mathbf{m}}$ is cocyclic with respect to the action of $(\mathfrak{n}^-)^a$ with a cocyclic vector

$$v_{\mathbf{m}}^* = \prod_{i \leq j} (X_{1, \dots, i}^{(i,j)})^{m_{i,j}}.$$

Let \mathbf{m} be a collection such that for some i, j we have $m_{i,j} = 1$ and $m_{c,d} = 0$ for all $(c, d) \neq (i, j)$ (these are analogues of the fundamental weights). We denote the corresponding homogeneous component $Q_{\mathbf{m}}$ by $Q_{i,j}$.

We first define the structure of $(\mathfrak{n}^-)^a$ -module on $Q_{i,j}$. Note that the space $Q_{i,j}$ has a basis $X_L^{(i,j)}$ with $L = (1 \leq l_1 < \dots < l_i \leq n)$ and $L \subset \{1, \dots, i, j+1, \dots, n\}$. We identify $Q_{i,j}$ with the dual space $\Lambda^i(W_{i,j})$ by setting $X_L^{(i,j)} w_P = \delta_{L,P}$ for any $P = (1 \leq p_1 < \dots < p_i \leq n)$, where

$w_P = w_{p_1} \wedge \cdots \wedge w_{p_i}$. This endows $Q_{i,j}$ with the structure of \mathfrak{g}^a -module, $Q_{i,j}^* \simeq \Lambda^i(W_{i,j})$. In particular, one has

$$f_{a,b}X_L^{(i,j)} = \begin{cases} -X_{l_1, \dots, l_{r-1}, a, l_{r+1}, \dots, l_i}^{(i,j)}, & \text{if } a \leq i \leq j < b \text{ and } l_r = b+1, \\ 0, & \text{otherwise.} \end{cases}$$

(Recall that for $\sigma \in S_i$ we have $X_{\sigma L} = (-1)^\sigma X_L$). We note that $Q_{i,j}$ is cocyclic with respect to the action of $(\mathfrak{n}^-)^a$ and $X_{1, \dots, i}^{(i,j)}$ is a cocyclic vector.

Now, let us define the structure of $(\mathfrak{n}^-)^a$ -module on all $Q_{\mathbf{m}}$. By definition, there exists a surjective map $\bigotimes_{i \leq j} Q_{i,j}^{\otimes m_{i,j}} \rightarrow Q_{\mathbf{m}}$. We have a natural structure of $(\mathfrak{n}^-)^a$ -module on the tensor product and hence we only need to show that $I(R_n)$ is $(\mathfrak{n}^-)^a$ -invariant. We first prove this for J_n .

Lemma 3.9. *The ideal J_n is invariant with respect to the $(\mathfrak{n}^-)^a$ -action defined above.*

Proof. It suffices to prove that for any element $f_{a,b} \in (\mathfrak{n}^-)^a$ one has

$$f_{a,b}R_{L,P}^{(i_1, j_1), (i_2, j_2); k} \in J_n.$$

But it is easy to see that $f_{a,b}R_{L,P}^{(i_1, j_1), (i_2, j_2); k}$ is again (different) generalized Plücker relation from our list. \square

Theorem 3.10. *The ideal $I(R_n)$ is $(\mathfrak{n}^-)^a$ -invariant and it is a saturation of J_n .*

Proof. We first prove the second statement. Using Proposition 3.5 we find numbers $N_{i,j}$ and a polynomial G such that

$$(3.3) \quad \prod_{i \leq j} (X_{1, \dots, i}^{(i,j)})^{N_{i,j}} F - G \in J_n$$

and G depends on the variables of PBW-degree at most 1. In addition, only one variable of the form $X_{1, \dots, r-1, r+1, \dots, i, m}^{(i,j)}$ shows up in G for any pair $r < m$. We claim that if $F \in I(R_n)$ then $G = 0$. In fact, we choose pairs r, m and i, j with $r \leq i \leq j < m$ such that G depends on $X_{1, \dots, r-1, r+1, \dots, a, m}^{(i,j)}$. Let

$$G = \sum_{p \geq 0} \left(X_{1, \dots, r-1, r+1, \dots, i, m}^{(i,j)} \right)^p G_p$$

be the decomposition with respect to powers of $X_{1, \dots, r-1, r+1, \dots, i, m}^{(i,j)}$ (i.e. G_p are independent of $X_{1, \dots, r-1, r+1, \dots, i, m}^{(i,j)}$). Proposition 3.6 gives that $\Psi G = 0$. We note that $\Psi X_{1, \dots, r-1, r+1, \dots, i, m}^{(i,j)} = (-1)^{i-r} T_{i,j} Z_{r, m-1}$. Therefore, the variable $Z_{r, m-1}$ comes only from $X_{1, \dots, r-1, r+1, \dots, i, m}^{(i,j)}$. We conclude that all $G_p = 0$. Continuing this procedure, we arrive at $G = 0$. Therefore, (3.3) now reads as

$$\prod_{i \leq j} (X_{1, \dots, i}^{(i,j)})^{N_{i,j}} F \in J_n$$

and thus $I(R_n)$ is the saturation of J_n (see [H], Exercise 5.10).

In order to prove the first statement of the proposition we note that for all a, b, i, j one has $f_{a,b}X_{1,\dots,i}^{(i,j)} = 0$. Since J_n is \mathfrak{g}^a -invariant, we obtain

$$\prod_{i \leq j} (X_{1,\dots,i}^{(i,j)})^{N_{i,j}} (f_{a,b}F) \in J_n \subset I(R_n).$$

Now since $I(R_n)$ is simple, we obtain $f_{a,b}F \in I(R_n)$. \square

Thanks to the theorem above, each $Q_{\mathbf{m}}$ carries the structure of $(\mathfrak{n}^-)^a$ -module. Our next goal is to prove that it is cocyclic with cocyclic vector being the product of $X_{1,\dots,i}^{(i,j)}$, i.e for any nontrivial $F \in Q_{\mathbf{m}}$ one has

$$\prod_{i \leq j} (X_{1,\dots,i}^{(i,j)})^{m_{i,j}} \in \mathbb{C}[f_{a,b}]_{a \leq b} F.$$

Theorem 3.11. *$Q_{\mathbf{m}}$ is cocyclic with a cocyclic vector $\prod_{i \leq j} (X_{1,\dots,i}^{(i,j)})^{m_{i,j}}$.*

Proof. Let us denote by $v_{\mathbf{m}}^*$ the image of $\prod_{i \leq j} (X_{1,\dots,i}^{(i,j)})^{m_{i,j}}$ in Q_n . Let $F \in \mathbb{C}\mathbf{X}_n$ and $F \notin I(R_n)$. We use Proposition 3.5 and write

$$\prod_{i \leq j} (X_{1,\dots,i}^{(i,j)})^{N_{i,j}} F - G \in J_n$$

for some numbers $N_{i,j}$ and a polynomial G . Since $F \notin I(R_n)$ we have $G \neq 0$. Suppose that a term

$$\prod_{i \leq j} (X_{1,\dots,i}^{(i,j)})^{l_{i,j}} \prod_{r < m} (X_{1,\dots,r-1,r+1,\dots,i(r,m),m}^{(i(r,m),j(r,m))})^{p(r,m)}$$

with some indices $i(r,m) \leq j(r,m)$ and powers $p(r,m)$ shows up in G with a non-zero coefficient. Then since $f_{a,b}X_{1,\dots,i}^{(i,j)} = 0$, we have

$$\prod_{r < m} f_{r,m}^{p(r,m)} G = \prod_{i \leq j} (X_{1,\dots,i}^{(i,j)})^{l_{i,j}} \prod_{r < m} f_{r,m}^{p(r,m)} (X_{1,\dots,r-1,r+1,\dots,i(r,m),m}^{(i(r,m),j(r,m))})^{p(r,m)}$$

and the result is proportional (with a non-zero coefficient) to the product of powers of $X_{1,\dots,i}^{(i,j)}$. Hence we arrive at the following identity: there exist numbers $N_{i,j}$, $p(r,m)$ such that

$$\prod_{r < m} f_{r,m}^{p(r,m)} \left(\prod_{i \leq j} (X_{1,\dots,i}^{(i,j)})^{N_{i,j}} F \right) = \text{const} \cdot \prod_{i \leq j} (X_{1,\dots,i}^{(i,j)})^{N_{i,j} + m_{i,j}}.$$

Since all $f_{a,b}$ annihilate $X_{1,\dots,i}^{(i,j)}$ we obtain

$$\prod_{i \leq j} (X_{1,\dots,i}^{(i,j)})^{N_{i,j}} \prod_{r < m} f_{r,m}^{p(r,m)} F - \prod_{i \leq j} (X_{1,\dots,i}^{(i,j)})^{N_{i,j} + m_{i,j}} \in I(R_n).$$

This gives

$$\left(\prod_{r < m} f_{r,m}^{p(r,m)} \right) F - \prod_{i \leq j} (X_{1,\dots,i}^{(i,j)})^{m_{i,j}} \in I(R_n)$$

and hence $(\prod_{r < m} f_{r,m}^{p(r,m)})F = v_{\mathbf{m}}^*$ in Q_n . \square

Corollary 3.12. *The modules $Q_{\mathbf{m}}$ are cocyclic with respect to the abelian algebra $(\mathfrak{n}^-)^a$ with cocyclic vectors $v_{\mathbf{m}}^*$. One has the isomorphism of $(\mathfrak{n}^-)^a$ -modules $Q_{\mathbf{m}}^* \simeq M_{\mathbf{m}}$.*

Proof. It is easy to see that the $(\mathfrak{n}^-)^a$ -modules $Q_{i,j}^*$ and $M_{i,j}$ are isomorphic. Since both families $Q_{\mathbf{m}}^*$ and $M_{\mathbf{m}}$ can be constructed inductively via the $(\mathfrak{n}^-)^a$ -embeddings

$$M_{\mathbf{m}^1 + \mathbf{m}^2} \subset M_{\mathbf{m}^1} \otimes M_{\mathbf{m}^2}, \quad Q_{\mathbf{m}^1 + \mathbf{m}^2}^* \subset Q_{\mathbf{m}^1}^* \otimes Q_{\mathbf{m}^2}^*,$$

our corollary follows. \square

4. CONJECTURES AND FURTHER DIRECTIONS.

In this section we collect conjectures concerning geometric and algebraic objects described in the present paper.

For the readers convenience we first recall the conjecture about the Plücker relations already stated in the previous section. Recall the ideal J_n generated by the (generalized) Plücker relations.

Conjecture 4.1. The ideals J_n and $I(R_n)$ coincide.

One way to prove such kind of statements is to find a set of linearly independent monomials in $Q_n = \mathbb{C}\mathbf{X}_n/I(R_n)$ such that these monomials form a spanning set for $\mathbb{C}\mathbf{X}_n/J_n$ (recall $J_n \subset I(R_n)$). This is usually done in terms of some kind of (semi)standard tableaux (see [Fu], [Fe1]). It is interesting to describe the corresponding combinatorics in our situation.

The following conjecture provides a presentation of the modules $M_{\mathbf{m}}$ in terms of generators and relations. It generalizes similar statements from [FFoL1], where the case of \mathbf{m} supported on the diagonal was considered ($m_{i,j} = 0$ unless $i = j$).

Conjecture 4.2. The following \mathfrak{g}^a -module $M_{\mathbf{m}}$ is isomorphic to the quotient $\mathbb{C}[f_{i,j}]_{i \leq j}/I_{\mathbf{m}}$, where $I_{\mathbf{m}}$ is the ideal generated by the subspace

$$(4.1) \quad \mathbf{U}(\mathfrak{b}) \circ \text{span}(f_{i,j}^{\sum_{i \leq k \leq l \leq j} m_{k,l} + 1}, i \leq j).$$

We also conjecture that the monomial basis of [FFoL1], [V] can be extended to our case. Namely, let $S_{\mathbf{m}}$ be the subset of $\mathbb{Z}_{\geq 0}^{n(n-1)/2}$ consisting of collections $\mathbf{s} = (s_{\alpha})_{\alpha > 0}$ such that for any Dyck path \mathbf{p} starting at α_i and ending at α_j one has

$$\sum_{\beta \in \mathbf{p}} s_{\beta} \leq \sum_{i \leq k \leq l \leq j} m_{k,l}.$$

Conjecture 4.3. The elements $\{f^{\mathbf{s}}v_{\mathbf{m}}, \mathbf{s} \in S_{\mathbf{m}}\}$ form a basis of $M_{\mathbf{m}}$.

We note that two conjectures above are closely related. Namely, it is easy to show that there is a surjection $\mathbb{C}[f_{i,j}]_{i \leq j}/I_{\mathbf{m}} \rightarrow M_{\mathbf{m}}$ (because the relations of the left hand side hold in $M_{\mathbf{m}}$). In addition we have the following lemma:

Lemma 4.4. *The vectors $\{f^{\mathbf{s}}, \mathbf{s} \in S_{\mathbf{m}}\}$ span the right hand side of (4.1).*

Proof. The proof is very similar to the one in [FFoL1]. □

Hence in order to prove two conjectures above it suffices to show that the vectors $\{f^{\mathbf{s}}v_{\mathbf{m}}, \mathbf{s} \in S_{\mathbf{m}}\}$ are linearly independent. Computer experiments support this conjecture, but we are not able to prove it so far. The main difference with the case of [FFoL1] is that the the Minkowsky sum $S_{\mathbf{m}^1} + S_{\mathbf{m}^2}$ is no longer equal to $S_{\mathbf{m}^1 + \mathbf{m}^2}$ for general \mathbf{m}^1 and \mathbf{m}^2 .

ACKNOWLEDGMENTS

This work was partially supported by the Russian President Grant MK-3312.2012.1, by the Dynasty Foundation and by the AG Laboratory HSE, RF government grant, ag. 11.G34.31.0023. This study comprises research findings from the ‘Representation Theory in Geometry and in Mathematical Physics’ carried out within The National Research University Higher School of Economics’ Academic Fund Program in 2012, grant No 12-05-0014. This study was carried out within The National Research University Higher School of Economics Academic Fund Program in 2012-2013, research grant No. 11-01-0017.

REFERENCES

- [A] I.Arzhantsev, *Flag varieties as equivariant compactifications of \mathbb{G}_a^n* , Proc. Amer. Math. Soc. 139 (2011), no. 3, 783–786.
- [Fe1] E.Feigin, \mathbb{G}_a^M *degeneration of flag varieties*, arXiv:1007.0646, to appear in Selecta Mathematica.
- [Fe2] E.Feigin, *Degenerate flag varieties and the median Genocchi numbers*, Math. Res. Lett. 18 (2011), no. 06, pp. 116.
- [FF] E.Feigin and M.Finkelberg, *Degenerate flag varieties of type A: Frobenius splitting and BWB theorem*, arXiv:1103.1491.
- [FFiL] E.Feigin, M.Finkelberg, P.Littelmann, *Symplectic degenerate flag varieties*, arXiv:1106.1399.
- [FH] W.Fulton, J.Harris, (1991), *Representation theory. A first course*, Graduate Texts in Mathematics, Readings in Mathematics **129**, New York.
- [FFoL1] E.Feigin, G.Fourier, P.Littelmann, *PBW filtration and bases for irreducible modules in type A_n* , Transformation Groups **16**, Number 1 (2011), 71–89.
- [FFoL2] E. Feigin, G. Fourier, P. Littelmann, *PBW filtration and bases for symplectic Lie algebras*, International Mathematics Research Notices 2011; doi: 10.1093/imrn/rnr014.
- [Fu] W.Fulton, *Young Tableaux, with Applications to Representation Theory and Geometry*. Cambridge University Press, 1997.
- [H] R.Hartshorne, *Algebraic Geometry*. GTM, No. 52. Springer-Verlag, 1977.
- [HT] B.Hassett, Yu.Tschinkel, *Geometry of equivariant compactifications of \mathbb{G}_a^n* , Int. Math. Res. Notices 20 (1999), 1211–1230.

- [K] S.Kumar, *Kac-Moody Groups, Their Flag Varieties and Representation Theory*, Progress in Mathematics, Vol. 204, Birkhäuser, Boston (2002).
- [PY] D.Panyushev, O.Yakimova, *A remarkable contraction of semisimple Lie algebras*, arXiv:1107.0702.
- [V] E.Vinberg, *On some canonical bases of representation spaces of simple Lie algebras*, conference talk, Bielefeld, 2005.
- [Y] O.Yakimova, *One-parameter contractions of Lie-Poisson brackets*, arXiv:1202.3009.

EVGENY FEIGIN:

DEPARTMENT OF MATHEMATICS,
NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS,
VAVILOVA STR. 7, 117312, MOSCOW, RUSSIA,
and

TAMM THEORY DIVISION, LEBEDEV PHYSICS INSTITUTE
E-mail address: `evgfeig@gmail.com`